Calculus: Differentiation
Proof of the Product Rule

Suitable for Senior Cycle

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THIS IS A HEA FUNDED CPD PROJECT WITH EPI•STEM:

Differentiation by Product Rule

If we have a function that is the product of two (or more) functions we use the product rule (product means multiply):

The product rule is defined as:

$$f(x) = u.v$$

$$f'(x) = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

Proof of the Product Rule

To prove this rule you will need the following:

- Differentiation from first principles formula
- Property 1 & 4 of Limit Laws as per Formulae & Tables

Limit Law in symbols	Limit Law in words
$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$	The limit of a sum is equal to the sum of the limits.
$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)]$	The limit of a product is equal to the product of the limits.

- And a little trick!

Proof of the Product Rule

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$$\frac{d}{dx}(\mathrm{f}(\mathrm{x})g(x)) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

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We subtract and add the term $f(x+h)g(x)$ to get:

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Take out factors:

$$\frac{d}{dx}(\mathrm{f}(\mathrm{x})g(x)) = \lim_{h\to 0}\frac{f(x+h)\,[g(x+h)-g(x)]+g(x)[f(x+h)-f(x)]}{h}$$

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Using Limit Properties we can then do the following:

$$\frac{d}{dx}(\mathrm{f(x)}g(x)) = \lim_{h\to 0} f(x+h)\ \lim_{h\to 0}\frac{(g(x+h)-g(x))}{h} + \lim_{h\to 0} g(x)\ \lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}$$

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Using Limit Properties we can then do the following:

$$\frac{d}{dx}(\mathrm{f(x)}g(x)) = \lim_{h\to 0} f(x+h)\ \lim_{h\to 0}\frac{(g(x+h)-g(x)}{h}+\lim_{h\to 0} g(x)\ \lim_{h\to 0}\frac{(f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}(\mathbf{f(x)}\ g(x)) = f(x)g'(x)+g(x)f'(x)$$

Now it’s your turn…

Using the approach taken in this video, could you now have a go at proving:

- The Quotient Rule

- The Chain Rule

Bibliography

- State Examinations Commission. (2014) *Formulae and Tables.* Dublin: The Stationary Office

Contact Details

Register for on-line CPD resources: https://epistem.ie

EPI•STEM project: Resources

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This on-line CPD project [HEA funded] is an initiative with EPI•STEM for science and mathematics secondary teachers in Ireland. The research-led development team include:

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